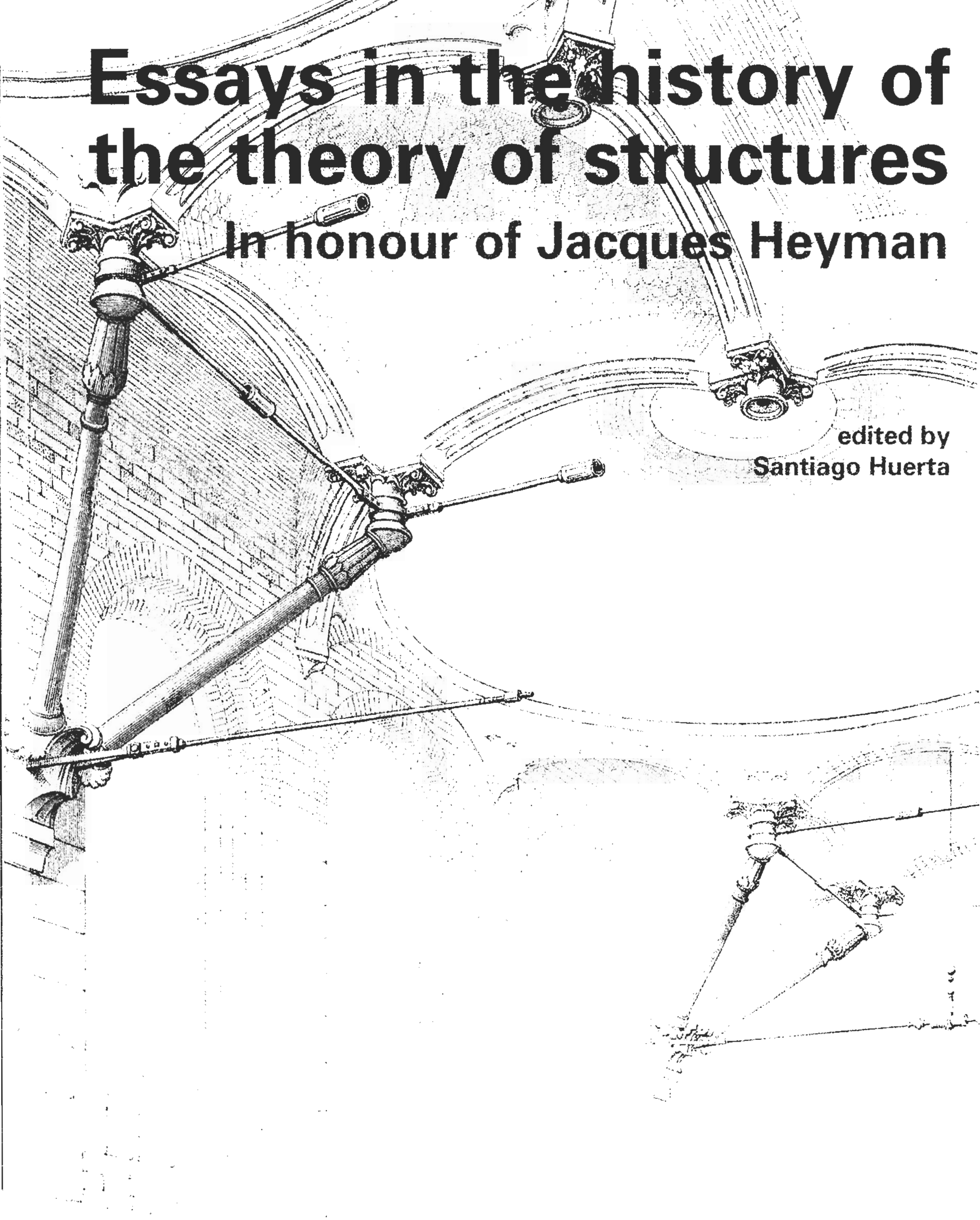
Essays in the history of the theory of structures

In honour of Jacques Heyman



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On the safety of the masonry arch. Different formulations from the history of structural mechanics

Federico Foce

A decalogue of false hypotheses

In 1927 American engineer George Fillmore Swain published a noteworthy treatise of *Structural Engineering* whose third volume, devoted to *Stresses, graphical statics and masonry*, contains a whole chapter on the stone arch that deserves great attention. In the footnote at the beginning of the chapter Swain advises the reader with these words:

> Since the stone arch is an elastic arch, differing only in degree from a monolithic concrete arch, it is impossible to distinguish sharply between the two. Before the student of structural engineering begins the study of elastic arches, it is desirable that he should study carefully this chapter on the stone arch, notwithstanding the fact that stone voussoir arches are now seldom built (Swain 1927, 400).

This warning is meaningful. Behind it we read an old attitude reflecting the distinction, also in terms of mechanical behaviour, between *constructions en maçonnerie* and *constructions en charpente,* a *Leitmotiv* of the technical literature on the *science des ingénieurs* since its beginning. At the date of publication of Swain's treatise this attitude was "officially" banished from the studies of structural engineering for a very obvious reason: it was unanimously accepted that stone, as well as steel or wood, are characterized by the common properties of strength and elasticity, even though with different degree and specific features. Thus, the elastic methods used for steel and wood structures can be reasonably applied to stone and masonry constructions.

This is exactly why Swain's words are of special interest. He was a distinguished scholar of elasticity and strength of materials and, as a young student, he had spent some time in Berlin studying theory of elastic systems under Emil Winkler, a pioneer in the field of the *Elastizitätslehre* and one of the first authors to support the application of the elastic analysis to the stone arch (Winkler 1879). Despite this scientific training, Swain's view stands out from the chorus and probably represents the best methodological lesson from a professional elastician. He writes:

> the elastic theory seems to be firmly intrenched in American engineering literature. Perhaps some who use it do not realize its defects and assumptions, and like it because it is complex and mathematical. It seems to be a curious characteristic of the human mind that it so often prefers complexity to simplicity, and mistakes obscurity for profundity . . . The writer believes in elastic methods, if they are necessary; not if they are unnecessary and if a simpler method is just as good (Swain 1927, 424).

This explicit position against any acritical application of the elastic theory to the masonry arch is accompanied by a decalogue listing the theoretical hypotheses assumed in the elastic analysis and, in parentheses, a brief remark on the real state of things:

The elastic theory is often termed "exact". The assumptions made in it are the following.

1. That the ends are rigid and do not rotate (this is untrue).
2. That the span does not change at all (this is untrue).
3. That the material is homogeneous (this is untrue).
4. That the modulus of elasticity is constant, not changing with the pressure (this is untrue, though perhaps close).
5. That the terms with r in the denominator may be neglected (this may be far from true).
6. That the integrals may be replaced by summations (this is approximate).
7. That the formulae for flexure are exact (this is untrue).
8. The stresses due shrinkage are neglected.
9. That the section is a rectangle (this is untrue; see Art. 18).
10. That the loads may be determined accurately (this is untrue; both the loads and their distribution on the arch are quite uncertain).

Possibly to some minds, not too mathematical, these facts may justify some of the conclusions in the present chapter (Swain 1927, 425).

These conclusions, and the "simpler method" that Swain refers to in the previous quotation, can be summarized with the following arguments, based on a particular use of Winkler's theorem on the minimum of the deformation work. As known, this theorem states that for an arch of constant section, under vertical loads, the true line of thrust is approximately the one which lies nearest the axis of the arch ring, in the sense that the sum of the squares of the vertical deviations is a minimum. Starting from this theorem, Swain observes that if we can draw within the arch ring the lines of minimum and maximum thrust,

> it is reasonably certain that there is a line nearer the center line than either of these, and that the arch is stable. The writer therefore believes the statement to be true that *if any line of resistance* (that is a line of thrust) *can be drawn within the arch ring the arch is stable*; and if a line of resistance can be drawn within the middle third, the true one will also be within the middle third, and there will be compression over the whole of every joint (Swain 1927, 412).

Swain is perfectly aware that this way of applying Winkler's theorem does not give the true line of thrust. To locate it, it should be necessary to find the line that actually minimizes the sum of the squares of the vertical deviations, as Baker remarked polemically. But, Swain replies,

> it is not necessary to do all this. It is only necessary to observe that, starting with the maximum line and gradually reducing the thrust and raising the point of application in the crown, the line will gradually change from the maximum to the minimum line, and that surely there will be some line that will be nearer the center line than either maximum or minimum, in order to conclude that if any line can be drawn in the arch which is not at the same time maximum and minimum, the true line will be in the arch and the arch is stable (Swain 1927, 414).

Obviously, Swain continues,

> if it were necessary to find accurately the true line, it would be necessary to do what Professor Baker says; and of course the *stresses* at the edges of a joint cannot be found accurately unless the true line is found. But this is unnecessary to judge *stability*. The stresses can be computed with quite sufficient accuracy without doing all this. The writer considers it a useless expenditure of time to try to find the true line anyway, considering the many uncertainties of the problem, regarding loads, their distribution, the material and workmanship, and even the so-called accurate elastic theory itself (Swain 1927, 414)

To conclude, Swain suggests that the stone arch, as well as the plain or reinforced concrete arch, can be studied with complete reliance on the results, if the followings are admitted:

1. The true line of resistance is one lying nearest the center line.
2. As we pass gradually from the maximum to the minimum line, some line is sure to be found which is nearer the center line than either the so-called maximum or the minimum.
3. It is not necessary to compute the stresses at the edges of the joints with extreme exactness (Swain 1927, 424).

It is easy to understand that these three points and of Swain's arguments appear somewhat familiar if we look at them from the viewpoint of modern limit analysis. Points 1 and 2 state, on the whole, that if we can draw within the arch the lines of maximum and minimum thrust, then the arch is stable because, in accordance with Winkler's theorem, the true line will also be within the arch. Point 3 clearly asserts that we do not need to find the true line of thrust, because the actual stresses cannot be computed with exactness. In other words, local strength is of secondary importance if global stability is ascertained. Even though Swain's deductive reasoning is based on Winkler's theorem, which is a theorem of elasticity, his conclusions are perfectly correct. From our modern point of view, he gives the right answer to what Jacques Heyman, after the plastic "revolution" occurred in the theory of structures in the 1950s, was to call "Poleni's problem" in one of his most convincing paper on the methodology to be assumed in the case of the masonry arch (Heyman 1988). As a matter of fact Swain's attitude, despite his elastic-oriented education, is totally in tune with the old tradition of studies which Heyman himself first brought to light with his pioneering work on the limit analysis of the stone skeleton (Heyman 1966).

There is no need here to quote Heyman's well-known theoretical studies on the matter. It is sufficient to say that the mark left by these studies was indeed profound. Proof of this is found, on one hand, by the constant references made to them in the technical literature, and on the other hand, by the numerous translations and reproductions of his works in recent years. But this is not all. To Heyman goes the credit for having promoted a new research methodology that is still today largely outside the mainstream of the interests of the structural engineer. We are speaking of that line of research aiming at a critical reconstruction of the historical development of the structural disciplines by rereading the sources and comparing them with ideas, methods and knowledge of the present time, the same line of historical research pursued from the 1960s by Clifford Truesdell in the context of the mechanics of solids and materials and later followed by Edoardo Benvenuto and Salvatore di Pasquale in the field of structural mechanics. It is to these masters that the present paper intends to render homage.

Basic features for the collapse analysis of the masonry arch

Before discussing in some detail the principal eighteenth- and nineteenth-century contributions on the theory of the stone arch, we give here a brief presentation of the stability and collapse conditions of the arch modelled in accordance with Heyman's three well known hypotheses concerning masonry behaviour, that is: 1) masonry has no tensile strength; 2) masonry has an infinite compressive strength; 3) sliding failure cannot occur.

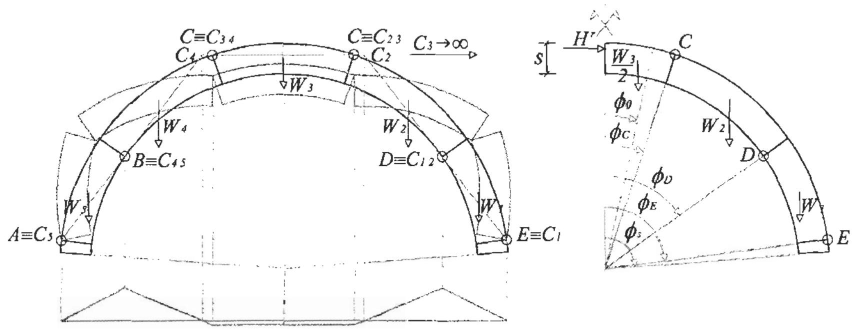

Figure 1
Collapse Mode I in its general form and corresponding kinematical chain

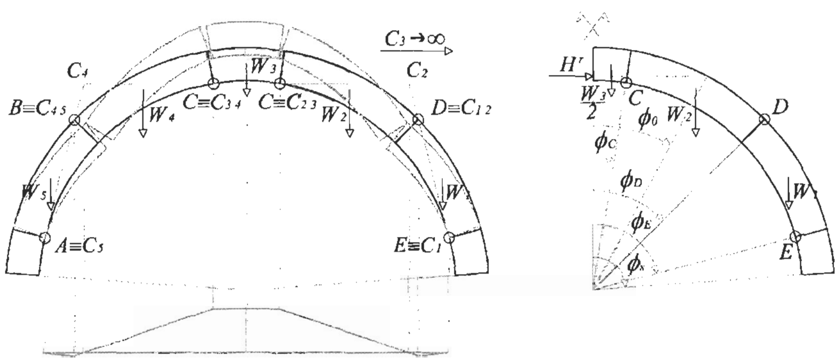

Figure 2
Collapse Mode II in its general form and corresponding kinematical chai

For the sake of simplicity, let us consider a symmetric arch of constant thickness s subject to a symmetric load. Two opposite rotational collapse modes with one degree of freedom may occur. Their general form is shown in figures 1 and 2 with the corresponding kinematic chain.

The angle ϕ_0 has been introduced in order to define the application point of the thrust H at the crown joint. In particular, if $\phi_0 = \phi_C = 0$ we find the two "usual" modes with hinge at the extrados or at the intrados of the crown joint, respectively (Figs. 3 and 4).

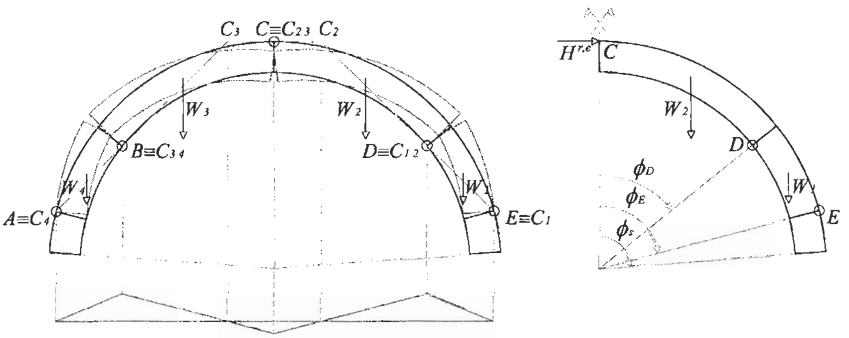

Figure 3
Collapse Mode I in the case of hinge at the crown extrados

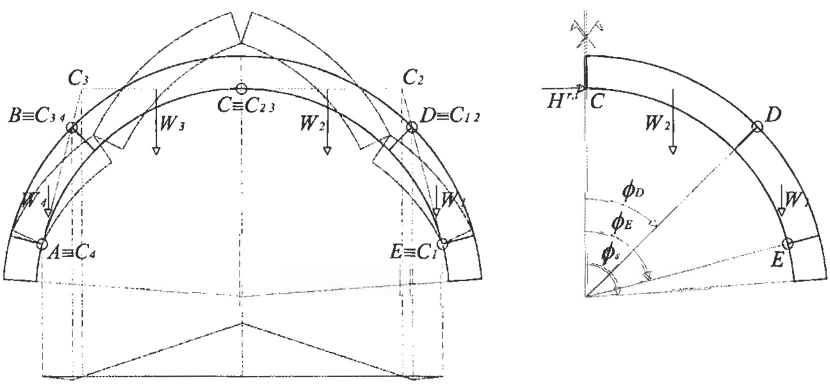

Figure 4
Collapse Mode II in the case of hinge at the crown intrados

The collapse analysis can be developed in terms of equilibrium equations or in terms of principle of virtual work. Whatever the approach, the collapse condition is the only one which is, at once, both statically and kinematically admissible with respect to a chosen collapse parameter, for instance the thickness of the arch.

Collapse analysis in terms of equilibrium equations

Let us consider a symmetric arch under a symmetric load and call $H^r_{min}(\phi, \phi_0, s)$ and $H^r_{max}(\phi, \phi_0, s)$ the values of horizontal thrust applied at a generic point of the crown for the equilibrium of a half arch about the intrados M and the extrados N of the joint at angle ϕ, respectively (Fig. 5). Given ϕ_0 and s, the first is a minimum, the latter a maximum.

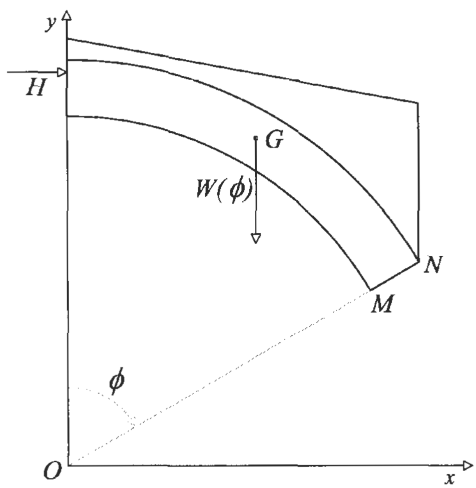

Figure 5

In order to avoid rotation about the intrados edge of any joint of the arch it must be

$$H \geq \max H^r_{min} \tag{1}$$

In order to avoid rotation about the extrados edge of any joint of the arch it must be

$$H \leq \min H^r_{max} \tag{2}$$

so that the necessary and sufficient condition for the equilibrium of the arch is

$$\max H^{r}_{\min} \le H \le \min H^{r}_{\max} \tag{3}$$

The necessary condition of collapse then becomes

$$\max H^{r}_{\min} = \min H^{r}_{\max} \text{ (statically admissible thrust)} \tag{4}$$

where $\max H^{r}_{\min} = H^{r}_{\min}(\phi_D)$ = and $\min H^{r}_{\max} = H^{r}_{\max}(\phi_C) = H^{r}_{\max}(\phi_E)$ for Mode I and $\min H^{r}_{\max} = H^{r}_{\max}(\phi_D)$ = and $\max H^{r}_{\min} = H^{r}_{\min}(\phi_C) = H^{r}_{\min}(\phi_E)$ for Mode II.

Condition (4) is also sufficient if the angles ϕ_C, ϕ_D and ϕ_E satisfy the disequalities

$$\phi_C < \phi_D < \phi_E \text{ (kinematically admissible mechanism).} \tag{5}$$

For thrust applied at the crown extrados (Mode I for $\phi_0 = \phi_C = 0$) (5) and (6) become

$$\max H^{r,e}_{\min} = \min H^{r,e}_{\max} \tag{7}$$

and

$$0 < \phi_D < \phi_E \tag{8}$$

where $\max H^{r,e}_{\min} = H^{r,e}_{\min}(\phi_D)$ and $\min H^{r,e}_{\max} = H^{r,e}_{\max}(\phi_E)$. Similarly, for thrust applied at the crown intrados (Mode II for $\phi_0 = \phi_C = 0$), (5) and (6) become

$$\max H^{r,i}_{\min} = \min H^{r,i}_{\max} \tag{9}$$

and

$$0 < \phi_D < \phi_E \tag{10}$$

where $\max H^{r,i}_{\min} = H^{r,i}_{\min}(\phi_E)$ and $\min H^{r,i}_{\max} = H^{r,i}_{\max}(\phi_D)$.

The previous analysis obviously admits a derivation in terms of line of thrust as well. Let us define first in a general way the properties of the so-called lines of minimum and maximum thrust for a symmetrical arch.

The line of minimum thrust is the steepest one possible within the arch ring, i. e., the most extended vertically and contracted horizontally; it necessarily touches the extrados at two symmetric points *e* near the crown (or at the extrados of the crown) and the intrados at two symmetric points *i* near the springings (or at the springings) (Fig. 6 a);

The line of maximum thrust is the flattest one possible within the arch ring, i. e., the most contracted vertically and extended horizontally; it necessarily touches the intrados at two symmetric points *i* near the crown (or at the intrados of the crown) and the extrados at two symmetric points *e* near the springings (usually at the springings) (Fig. 6 b);

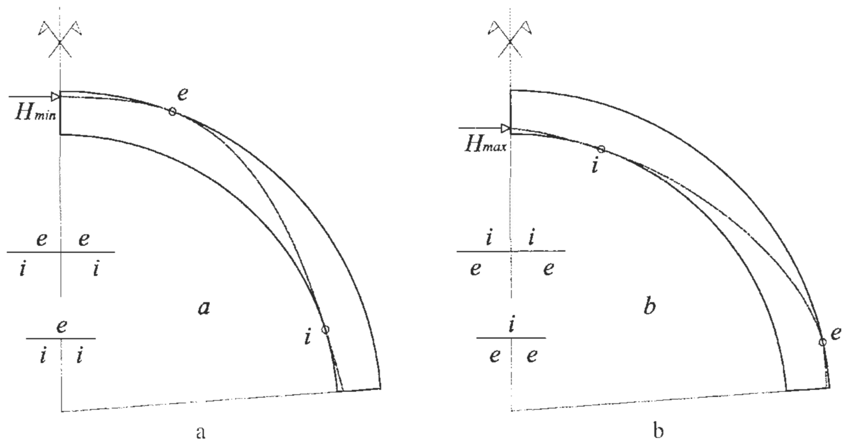

Figure 6
General form of the lines of minimum and maximum thrust for a symmetric arch

In terms of line of thrust, the collapse condition states that the arch fails only if the the lines of maximum and minimum thrust coincide, that is if only one line is possible and fulfils the condition for both maximum and minimum thrust.

Collapse analysis in terms of principle of virtual work

For rotational collapse modes the virtual work involves only the actives forces and is a function of the type $\delta L^{(a)}$ $(\phi_D, \phi_D, \phi_D, s)$. Obviously, the work function depends on the type of collapse mode, so that a function $\delta L_I^{(a)}$ for Mode I and a function $\delta L_{II}^{(a)}$ for Mode II must be defined. If, for a given thickness and for any compatible mechanisms, it is

$$\delta L^{(a)} < 0 \tag{11}$$

then the equilibrium is stable and collapse cannot occur. The collapse condition requires

$$\delta L^{(a)} = 0 \tag{12}$$

so that the collapse thickness can be determined by means of the two following theorems:

Static theorem: Among the set of the statically admissible states, the collapse thickness is the minimum thickness for which a kinematically admissible mechanism exists;

Kinematic theorem: Among the set of the kinematically admissible mechanisms, the collapse thickness is the maximum thickness for which a statically admissible state exists

Discussion of the principal pre-elastic historical theories on the collapse of the arch

In the section that follows a selection of historical works on the collapse of the arch is critically discussed and compared with the previous results. This selection inevitably neglects other important studies for which we refer to the bibliographies contained in recent works on the matter (Foce 2002, Kurrer 2002, Huerta 2004). However, it collects the main contributions to a general formulation of the collapse analysis of the arch from the point of view of the modern limit analysis based on Heyman's hypotheses. In this sense, our review of the historical sources will be exclusively focused on the rotational collapse modes, even though some of them take into account also sliding collapse modes in the presence of friction.

Coulomb (1773)

Coulomb's analysis in terms of method of maxima and minima is probably the first attempt at a general formulation of the collapse of symmetric arches. Supposing the thrust to be applied at a generic point of the crown and considering the rotational equilibrium of a voussoir, Coulomb correctly finds the necessary and sufficient condition of equilibrium (3). In the *Remarque I* of his *Essai* he adds that the horizontal thrust must act at the crown extrados "pour rendre la force B_1 [max H^r_{min} with our notations] aussi petite qu'elle puisse être" (Coulomb 1776, 380), so that he takes max $H^r_{min} \equiv$ max $H^{r,e}_{min}$. Nothing is said, however, about the application point of the thrust in order to compute the other extreme value min H^r_{max} and, consequently, to define the range of admissible thrusts. Two interpretations can be given of Coulomb's text: 1) he implicitly understands that the min H^r_{max} must be found with the thrust applied again at the crown extrados: in this case it should be min $H^r_{max} \equiv$ min $H^{r,e}_{max}$; 2) he implicitly supposes that the min H^r_{max} must be found with the thrust applied at the crown intrados, in order to take the greatest value of min H^r_{max}: in this case it should be min $H^r_{max} \equiv$ min $H^{r,i}_{max}$.

According to the first interpretation, Coulomb would give only (7), disregarding (8). The analysis of the rotational modes is correct as far as Mode I, but not complete in that Mode II is not considered. According to the second interpretation, which is the one usually assumed by the exegets of Coulomb's *Essai*, the equilibrium condition would be

$$\max H^{r,e}_{\min} \leq H \leq \min H^{r,i}_{\max} \tag{13}$$

If the range of thrust shrinks to a single value we have

$$\max H^{r,e}_{\min} = \min H^{r,i}_{\max} \tag{14}$$

Now, this equality cannot be the necessary condition of collapse because the two extreme values of the thrust are computed taking two different points of application at the crown joint. No rotational mode can occur, unless the crown joint is reduced to a single point, because only in this circumstance there would be no distinction between extrados and intrados. In this case the analysis of the rotational modes is incorrect, as pointed out by Persy (1825).

Mascheroni (1785)

Lorenzo Mascheroni is one of the few authors to have dealt with the equilibrium of the arch in terms of principle of virtual work. Before him a significant application of this principle to structural problems was given in 1743 by the "tre mattematici" in the known *Parere* on the stability of Saint Peter's dome. During the nineteenth century this approach was only occasionally adopted, and always without general purposes, for instance by Navier in a note to Gauthey's *Traité de la construction des ponts* (Gauthey 1809, 1, 318–320) and by Lambel (1822) in a memoir on the stability of arches and retaining walls. On the contrary, in Mascheroni's *Nuove ricerche* the principle of virtual work assumes a central role and is used programmatically for the equilibrium analysis of rigid systems with one degree of freedom.

In this sense Mascheroni's contribution ideally belongs to the old tradition of the "science of weights" of Aristotelian origin, according to which, as pointed out by Sinopoli (2002, 2003), the equilibrium condition of the "simple machines" —real mechanisms subject to weights— was intended as a condition of non-activated motion by stating that the work of those weights must be zero for a (virtual) vertical displacement of their center of mass. Under this point of view, when an arch fails and transforms into a mechanism, it becomes a particular "machine" whose equilibrium condition can be derived by means of the principle of virtual work. As this "machine" is formed by a certain number of voussoirs that have absolute and relative (infinitesimal) movements, the great difficulty consists in correctly describing these movements in order to apply the principle.

Mascheroni's theoretical contribution mainly concerns the solution of this kinematical problem. He first gives a general discussion of the infinitesimal displacement of a segment, then demonstrates kinematical theorems that are finally applied to solve the problem of the *Equilibrio de' Rettilinei*, that is of systems with one degree of freedom formed by rigid bars connected with hinges and subject to weights. On this general basis the problem of the arch is easely solved since, at collapse, the vertical dislacements of the voussoirs coincide with those of the bars connecting the hinges about which the voussoirs move. If G and Q are the centers of mass of the voussoirs or of the bars (Fig. 7), from the principle of virtual work Mascheroni derives the equilibrium condition for the modes I and II (Mascheroni 1785, 25)

$$G\left(\frac{BT}{AF}-\frac{CE}{BE}\right)-Q\frac{CK}{BE}=0 \tag{14}$$

which can represent both $\delta L_{I}^{(a)}=0$ and $\delta L_{II}^{(a)}=0$ provided that we invert the position of the hinges at the intrados and extrados lines.

Monasterio (ca.1800)

For the theory of the arch the unpublished manuscript *Nueva teórica sobre el empuje de bóvedas* by the Spanish engineer Joaquin Monasterio is noteworthy for several reasons, as recently shown (Huerta and Foce 2003). Tackling the problem of the arch with a quite general approach, Monasterio first develops an original way of deriving the kinematically admissible mechanisms by observing that, when an arch fails, the voussoirs have movements of rotation and translation characteristic of the various collapse modes. Thus, by defining *r* and *t* as the rotations and translations of the voussoirs, Monasterio describes the collapse modes by means of proper sequences of the letters *r* and *t*, (proper in the sense that they correspond to admissible mechanisms). The number of letters gives the number of voussoirs in which the arch breaks at collapse; the order of letters, from left to right, indicates the types of movement to which the voussoirs are subject. Monasterio applies this procedure to non-symmetrical arches and finds seven collapse mechanisms with one degree of freedom. They are described by the sequences *tt*, *rrr*, *rrt*, *trr*, *tr*, *rt*, *trt* and shown if figures 1–7 of his plate 1, where the two sequences *tr* and *rt* are collapse modes with a composed movement of rotation and translation at a certain joint (Fig. 8).

Monasterio's approach to the kinematics of the arch is new and promising indeed. However, his results are not fully satisfactory, as can be shown by a different way of forming the sequences. Let us identify as *R*, *T* and *RT* the joints (not the

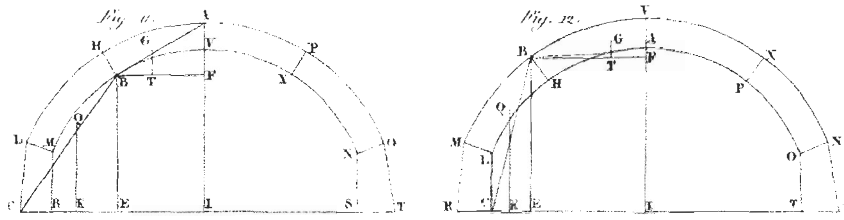

Figure 7
Sketches of the arches with the *rettilinei* for the analysis of Modes I and II (from Mascheroni 1785)

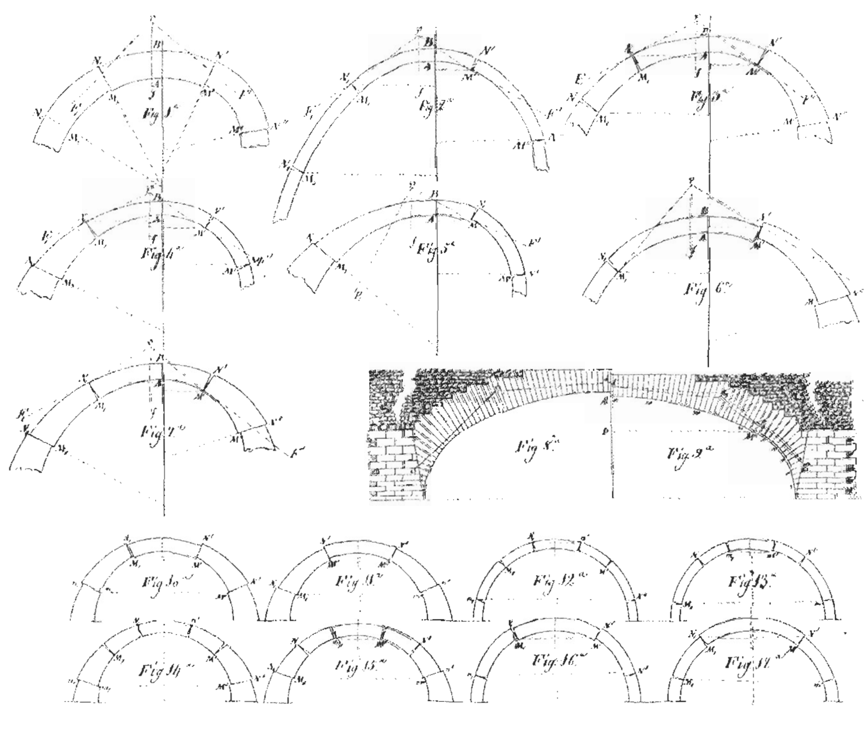

Figure 8
Monasterio's Plate 1 with the non-symmetric and symmetric collapse modes

voussoirs) where absolute and relative rotations, translations and roto-translations take place, so that the number of letters gives the number of rupture joints, and this number minus one gives the number of voussoirs at collapse. Through this choice it is easy to see that Monasterio's list is not complete because the rotation *r* or the translation *t* of a voussoir may derive from different types of absolute or relative movements *R*, *T* and *RT* at the rupture joints. As a matter of fact, there are twenty-three non-symmetrical collapse modes with one degree of freedom, represented by the following fourteen sequences, some of which stand for two opposite modes (Perazzo 2005):

T-T-T (corresponding to the sequence *tt*, figure 1 of Monasterio's plate 1)
T-R-R-T (corresponding to *trt*, figures 7 of plate 1)
T-T-R-R (two opposite modes)
R-R-R-T (two opposite modes) (valid fot both *rrt* and *trr* figure 3 and figure4 of plate 1)
R-R-R-R (corresponding to *rrr*, figure 2 of plate 1)
R-T-T-R
R-T-R-R (two opposite modes)
T-R-T-R (two opposite modes)
R-RT-R (two opposite modes)
RT-T-R (two opposite modes)
RT-R-R (two opposite modes)
RT-R-T (two opposite modes)
R-RT-T (two opposite modes) (valid for both *tr* and *rt*, figures 5 and 6 of plate 1)
RT-RT

For the symmetrical arch Monasterio does not write the sequences but provides the eight collapse modes drawn at the bottom of his plate 1. This number can rise to twelve if we consider that the modes of figures 12, 13, 14, 15, with two symmetrical hinges near the crown, may include four collapse modes with a single hinge at the extrados or intrados of the crown. Also in this case, however, Monasterio's list is not complete because there are twenty symmetrical modes with one degree of freedom, given by the following ten sequences representing two opposite modes (Perazzo 2005).

R-R-R-R-R (two opposite modes included in the modes of figures 12 and 13 of plate 1)
T-R-R-R-T (two opposite modes included in the modes of figures 14 and 15)
T-T-T-T (corresponding to figures 10 and 11)
R-T-R-T-R

R-RT-RT-R (corresponding to figures 16 and 17)
RT-R-R-RT
RT-R-RT
R-R-R-R-R-R (corresponding to figures 12 and 13)
T-R-R-R-R-T (corresponding to figures 14 and 15)
R-T-R-R-T-R

A second relevant feature of Monasterio's memoir comes from the fact that he tackles the stability analysis starting with the non-symmetrical arch and then adapting it to the special case of the symmetrical arch. Further, the way of deriving the collapse stability is proof of the originality of his analysis. For instance, the condition for the activation of the non-symmetrical mode *rrr* (*R-R-R-R* with our notation) is obtained by imposing necessary static requirements, that is (Fig. 9):

1) the right component of the weight must go through the intrados edge *C*;
2) the left component of the weight must go through the extrados edge *B*;
3) the moment of the left component of , with respect to the intrados edge *A* must be lower than the moment of the weight with respect to the same point;
4) the moment of the right component of with respect to the extrados edge *D* must be greater than the moment of the weight with respect to the same point.

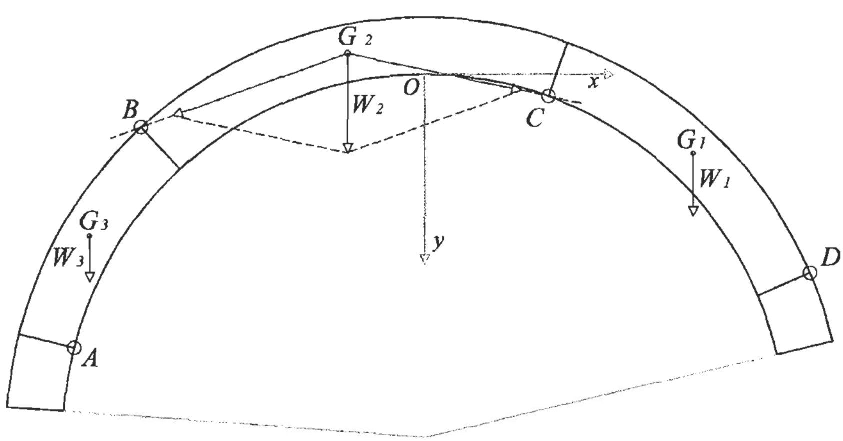

Figure 9
Monasterio's analysis of the rotational collapse mode of a non-symmetric arch (redrawn from Monasterio)

By treating these conditions Monasterio arrives at a stability disequality which he adapts to the symmetric arch. Thus, by taking the origin of the axes at the crown intrados, he correctly finds that collapse Mode I of figure 10 a cannot occur if (in our notations)

$$(W_1 + W_2)\frac{x_E - x_G}{y_E + s} - W_2\frac{x_D - x_{G_2}}{y_D + s} \geq 0 \quad (15)$$

and, similarly, that collapse Mode II of figure10b cannot occur if

$$W_2\frac{x_D - x_{G_2}}{y_D} - (W_1 + W_2)\frac{x_E - x_G}{y_E} \geq 0 \quad (16)$$

Now, the two terms in (15) and in (16) are nothing other than the thrusts $H^{r,e}_{max}$ and $H^{r,e}_{min}$ and the thrusts $H^{r,i}_{max}$ and $H^{r,i}_{min}$, respectively. Thus by searching for the minimum of the first terms of (15) and (16) and the maximum of the latter ones we obtain two stability disequalities which become (7) and (9) in the case of limit equilibrium.

Monasterio uses (15) for the semicircular arch of constant thickness under its own weight. By trial and error he finds that the minimum thickness is between 1/8 = 0.125 and 1/8 = 0.111 of the intrados radius and the rupture joint at the haunches is between 54° and 56° from the crown. This result is quantitatively correct and agrees with the calculation by Petit (1835), who gave 0.114. Mi-

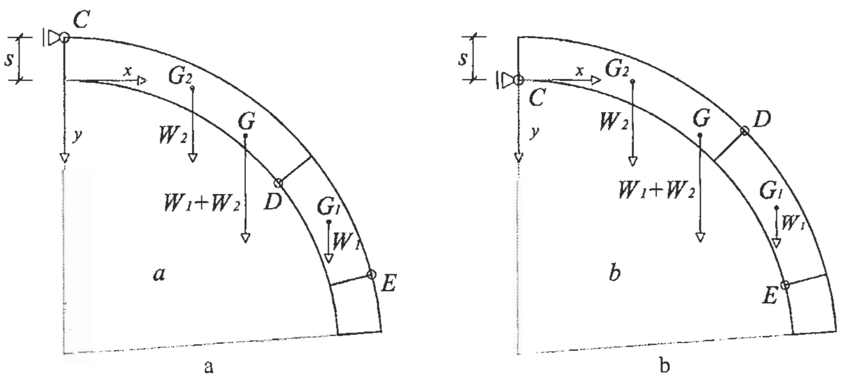

Figure 10
Monasterio's analysis of stability of a symmetric arch with respect to Modes I and II

lankovitch (1907) obtained the rigorous value 0.1136 corresponding to the rupture joint at 54°29' from the crown.

Persy (1825), Navier (1826), Michon (1857)

The contributions by Persy, Navier and Michon on the collapse of the arch are directly connected with Coulomb's method of maxima and minima and represent an important theoretical improvement of Coulomb's results. Other authors, both before and after them, have taken Coulomb's method as a point of departure, even though with less general purposes. In this sense we can cite the works of Berard (1810), Audoy (1820), Lamé and Clapeyron (1823), and the "special issue" of the *Mémorial de l'Officier du Génie* of 1835, where three long memoirs are devoted to particular applications of the method (Garidel and Petit) and to a graphic calculation of the extreme values of the thrust (Poncelet).

Persy's treatment of the matter is particularly enlightening because he intentionally starts from Coulomb's analysis to point out its deficiency regarding the point of application of the thrust at the crown joint. He initially considers the thrust applied at a generic point of the crown and finds the two extreme values max H^r_{min} and min H^r_{max}, to which he associates the joints *J* and *j*, respectively. To conceive a kinematically admissible mechanism when the necessary condition of collapse max H^r_{min} = min H^r_{max} is attained, he admits for a moment that the thickness at the crown shrinks to a single point and concludes that if the joint *J* is higher than *j*, the collapse mode of figure11a may occur and, conversely, if the joint *J* is lower than *j*, the collapse mode of figure 11 (b) may occur.

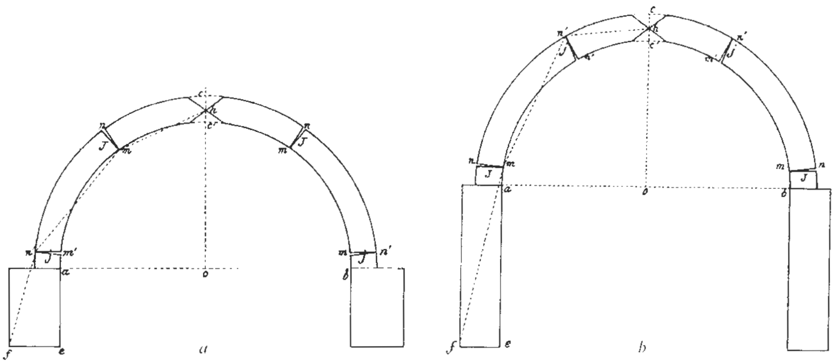

Figure 11
Persy's two rotational collapse modes in the hypothetical case of a point contact at the crown joint

Bearing in mind the two opposite rotational modes shown in figure 3 and figure 4 with a hinge at the crown, Persy observes that when the crown joint has a finite thickness, the point of application of the thrust may assume two limit positions corresponding to the crown extrados and the crown intrados. Thus, to obtain the collapse mode I of figure 3, with thrust at the crown extrados, he defines the new extreme values max $H^{r,e}_{\min}$ and min $H^{r,e}_{\max}$ and concludes that mode I may occur when max $H^{r,e}_{\min}$ = min $H^{r,e}_{\max}$ under the kinematical condition that the joint corresponding to max $H^{r,e}_{\min}$ is higher than the joint corresponding to $H^{r,e}_{\max}$; conversely, to obtain the collapse mode II of figure 4, with thrust at the crown intrados, he defines the new extreme values max $H^{r,i}_{\min}$ and min $H^{r,i}_{\max}$ and concludes that the mode II may occur when max $H^{r,i}_{\min}$ = min $H^{r,i}_{\max}$ under the condition that the joint corresponding to max $H^{r,i}_{\min}$ is lower than the joint corresponding to min $H^{r,i}_{\max}$.

Persy's analysis is correct in so far as collapse modes I and II shown in figure 3 and figure 4 are concerned. In this sense, Persy clarifies Coulomb's discussion since he introduces two pairs of extremes values of the thrust corresponding to the application point at the crown extrados and intrados and gives the collapse conditions (7) and (8) for mode I and (9) and (10) for mode II. These results were obtained independently by Navier (1826) and received a clear exposition in a later work by Michon (1857). Nevertheless they do not completely solve the collapse analysis of the symmetrical arch as they cannot include the general form of the rotational modes shown in figure1 and figure 2.

Durand-Claye (1867, 1868, 1880)

The progress of the studies on the elasticity and strength of materials during the first decades of the ninteteenth century had an indirect influence on the theory of the arch as well. Starting from the 1839s, the traditional approach in terms of collapse analysis soon appeared out of date in the face of new questions concerning the actual stresses within the arch and, for at least forty years, the problem of finding the "true" line of thrust seriously troubled the minds of generations of scholars before the elastic approach was recognized and accepted as the only rational way out. During these forty years many methods were proposed in order to remove the statical indetermination of the problem on the basis of arbitrarious assumptions regarding the point of application of the thrust at the crown or by means of metaphysical principles concerning the features of the actual line of thrust. The tragicomical result was what the Italian elastician Francesco Crotti denounced as a dizziness of the minds in a noteworthy *Esame critico* (Crotti 1875) written in reaction to the *n*th attempt at solving the problem by means of *a priori* hypotheses.

In this contradictory phase of the theory of the arch an important contribution was made in 1867 by Alfred Durand-Claye with the method of the areas of stability (Foce and Aita 2003). Durand-Claye was perfectly aware that the true thrust line of a stable arch is statically indeterminate. Thus, instead of searching for the "actual" thrust line, he elaborates a general method for determining all the admissible thrust lines which fulfill the equilibrium equations (with respect also to the strength of materials): in the case of a stable arch the true one will necessarily be included. As Durand-Claye himself writes, "de la possibilité de l'équilibre, nous concluons à la stabilité" (Durand-Claye 1867, 65).

Briefly, and taking into account only the part of the method dealing with the rotational equilibrium, Durand-Claye considers a voussoir of a symmetric arch and writes the equations of the thrusts $H^{r}_{\min}$ and $H^{r}_{\max}$, that is

$$H^{r}_{\min}(\phi, y) = \frac{W(\phi)[x_M(\phi) - x_G(\phi)]}{y - y_M(\phi)} \tag{17}$$

and

$$H^{r}_{\max}(\phi, y) = \frac{W(\phi)[x_N(\phi) - x_G(\phi)]}{y - y_N(\phi)} \tag{18}$$

where the vertical distance y is the independent variable defining the point of application of the thrust at the vertical joint and the differences of the coordinates are the lever arms of weight and thrust with respect to the intrados M and the extrados N of the joint at angle ϕ (Fig. 12 a). In the plane Hy these equations represent two equilateral hyperbolas with a common vertical asymptote coinciding with the line of the crown joint and with horizontal asymptotes given by the straight lines through the intrados M and the extrados N of the joint, respectively. Given ϕ, the admissible values of the crown thrust are graphically represented by the area bounded by the two hyperbolas and the two horizontal straight lines starting from the extrados and intrados edge of the crown joint. Now, by drawing the hyperbolas (17) and (18) for each joint of the arch and taking the common area to all the areas previously defined, Durand-Claye obtains the so-called *area of stability*, which in the plane Hy is the locus of the points representing the admissible values of the thrust and their point of application at the crown for the rotational equilibrium of the whole arch (Fig. 12 b).

If, for a certain value of the thickness, the area of stability shrinks to a point, then the collapse condition is attained since a unique admissible thrust exists and, at the same time, a certain collapse mode becomes kinematically admissible, with hinges located at the intrados or extrados of the joints corresponding to the intersecting hyperbolas. This case is shown in figure 13 (a) and corresponds to the collapse Mode I of a semicircular arch subject to its own weight and bearing a horizontal fill with the same specific weight as the arch,

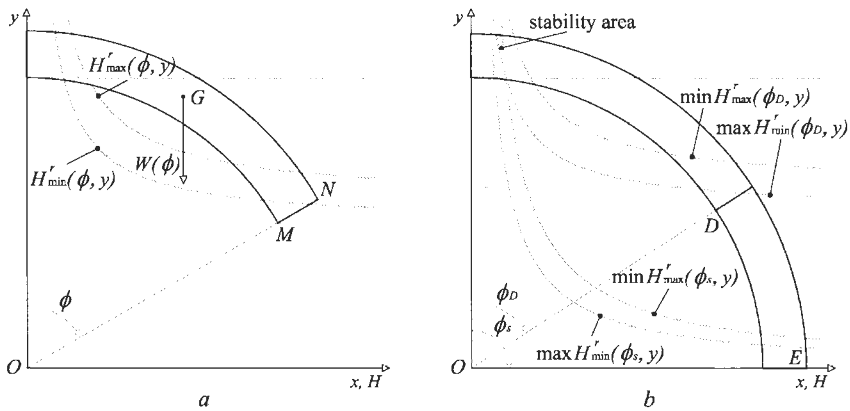

Figure 12
Area of the admissible thrusts for the rotational equilibrium of a generic voussoir (a) and area of stability for the whole arch (b)

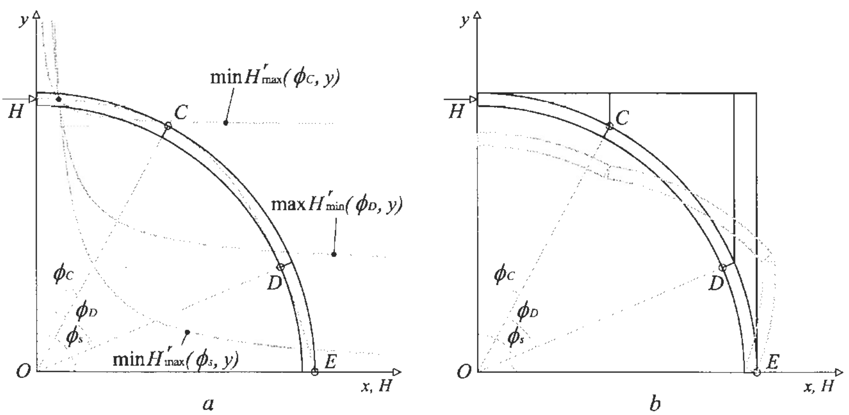

Figure 13
Collapse condition according to Durand-Claye's method (a) and corresponding collapse mode (b)

with thickness $s = 0.0481r$ and hinges at $\phi_C \cong 28°$, $\phi_D \cong 67°$ and at the springing (Fig. 13 b).

Scheffler (1857), Ceradini (1873, 1887)

Our interest in Scheffler's work concerns his study of the geometrical properties of the lines of maximum and minimum thrust, probably the first general analysis considering both symmetrical and non-symmetrical arches. Scheffler clearly recognizes that, for a stable arch, the statically admissible thrust lines are bounded by two limit lines corresponding to the minimum and maximum value of the horizontal thrust H and that the limit condition of equilibrium is attained when a line of thrust can be draw within the arch ring which has, at once, the property of the minimum and maximum thrust (Scheffler 1857, 48).

As far as the line of minimum thrust, Scheffler considers the thrust line through the extrados of the crown joint and the intrados of the springing and asserts that:

1) If this line lies within the arch ring, then it is the line of minimum thrust (Fig. 14 a);
2) If this line cuts the intrados, but not the extrados, then the line of minimum thrust goes through the extrados of the crown and touches the intrados at a certain point near the springing (Fig. 14 b);
3) If this line cuts the extrados, but not the intrados, then the line of minimum thrust touches the extrados at a certain point (starting from a point internal to the crown joint) and goes through the intrados of the springing (Fig. 14 c);
4) If this line cuts both the extrados and intrados, then the line of minimum thrust touches the intrados at a certain point (starting from a point internal to the crown joint) and touches the intrados at a certain point near the springing (Fig. 14 d);

Similarly, for the line of maximum thrust Scheffler considers the thrust line through the intrados of the crown joint and the extrados of the springing and asserts that:

1) If this line lies within the arch ring, then it is the line of maximum thrust (Fig. 15 a);
2) If this line cuts the intrados, but not the extrados, then the line of maximum thrust touches the intrados at a certain point (starting from a point internal to the crown joint) and goes through the extrados of the springing (Fig. 15 b);

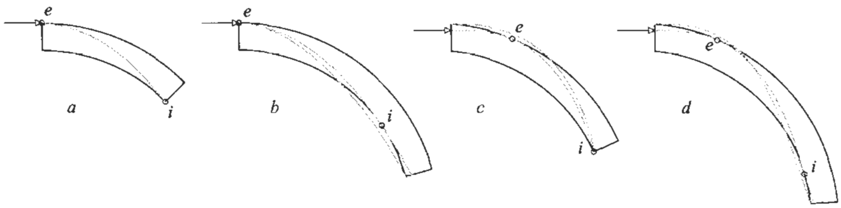

Figure 14
Possible positions of the line of minimum thrust (redrawn from Scheffler 1857)

3) If this line cuts the extrados, but not the intrados, then the line of maximum thrust goes through the intrados of the crown and touches the extrados at a certain point over the springing (Fig. 15 c);
4) If this line cuts both the extrados and intrados, then the line of maximum thrust touches the intrados at a certain point (starting from a point internal to the crown joint) and the extrados at a certain point near the springing (Fig. 15 d);

The previous discussion was given also by Ceradini (1873), who extended Scheffler's analysis on the basis of the following general statement: if two lines of thrust intersect, their points of intersection lie on the straight line connecting the point of intersection E of the two reactive systems at the left springing with the point of intersection F of the two reactive systems at the right springing (Fig. 16).

This statement is a direct consequence of the construction of two funicular polygons for the same external load. On its basis Ceradini shows all the possible relative positions of the thrust lines corresponding to different reactive systems and demonstrates the properties of the lines of maximum and minimum thrust for

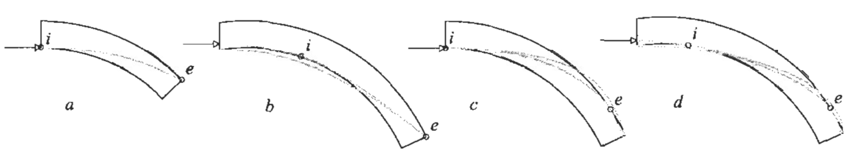

Figure 15
Possible positions of the line of maximum thrust (redrawn from Scheffler 1857)

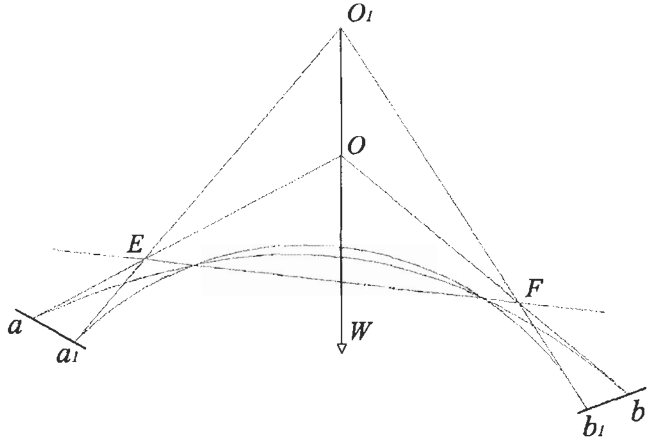

Figure 16
Ceradini's statement on the lines of thrust and the corresponding reactive systems (from Ceradini 1873)

a non-symmetrical arch under a vertical load. In particular, the line of minimum thrust necessarily touches the extrados at one point *e* and the extrados at two points *i* as in figure 17 (a), while the line of maximum thrust necessarily touches the intrados at one point *i* and the extrados at two points *e* as in figure 17 (b).

If a line of thrust has two points of contact with the extrados and two points of contact with the intrados located as in figure 18, it has the features of both the lines of maximum and minimum thrust and then it is the only statically admissible line.

Moreover, Ceradini shows that the same properties hold if the resultant of the external load is not vertical, with the difference that they refer to the component of the reactive systems which acts perpendicularly to that resultant (Ceradini 1887).

Conclusions

In Swain's treatise of 1927 no mention is made of the authors quoted above and is probable that their contributions were rather extraneous to his scientific training. In spite of that, it is not difficult to recognize a deep convergence between his approach and the results of the pre-elastic studies that we have discussed in the previous section. As a matter of fact, Swain's peroration in favour of a "weighted"

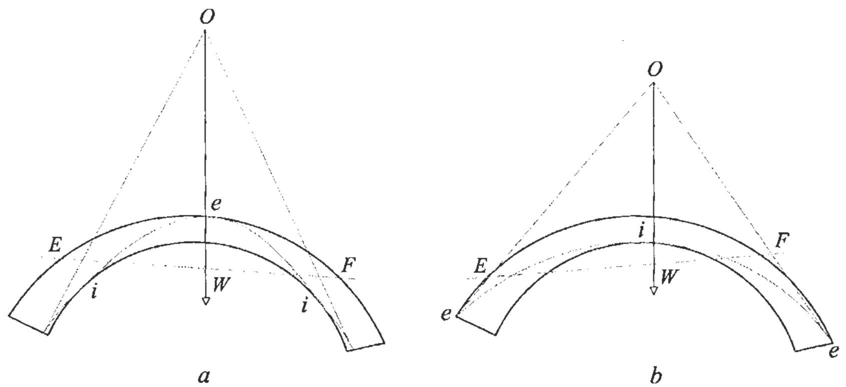

Figure 17
Lines of minimum and maximum thrust for a non-symmetrical arch (from Ceradini 1873)

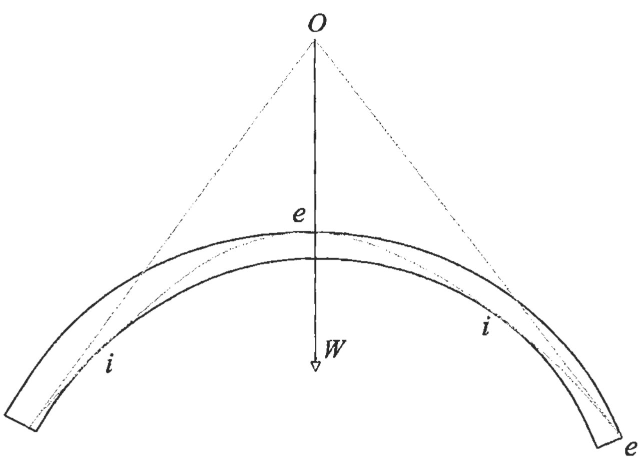

Figure 18
Line of minimum and maximum thrust for a non-symmetrical arch (from Ceradini 1873)

use of the elastic methods for the analysis of the masonry arch has the value of a methodological choice whose last consequences lead to the structural philosophy of limit analysis that, after Heyman's lesson, is nowadays considered as the basis for the study of the stone skeleton. In this sense, also the lesson from history should be carefully attended.

Acknowledgments

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